H.W. 3 Solution

Q1)

$$y' + y = xy^3;\ \text{Bernoulli}$$

$$y^{-3}y' + y^{-2} = x$$

$$\underbrace{\boxed{y^{-3}y'}}_{=v'} + (-2)\underbrace{\boxed{\frac{y^{-2}}{-2}}}_{=v} = x$$

$$\Longrightarrow v' - 2v = x \text{ Linear}$$

$$P(x) = -2 \Longrightarrow I(x) = e^{-2x}$$

$$\Longrightarrow e^{-2x}v' - 2e^{-2x}v = xe^{-2x}$$

$$\text{i.e. } (ve^{-2x})' = xe^{-2x}$$

$$\Longrightarrow ve^{-2x} = \int xe^{-2x}dx + C$$

$$= \int x(\frac{e^{-2x}}{-2})'dx + C$$

$$= x(\frac{e^{-2x}}{-2}) - \int (\frac{e^{-2x}}{-2}) \cdot 1dx + C$$

$$\Longrightarrow \frac{y^{-2}}{-2}e^{-2x} = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$

where C is an arbitrary constant.

Q2)

$$xy' + y = -2x^6y^4;\ \boxed{x > 0} \text{ Bernoulli}$$

$$y^{-4}y' + \frac{1}{x}y^{-3} = -2x^5$$

$$\underbrace{\boxed{y^{-4}y'}}_{=v'} + \frac{1}{x}(-3)\underbrace{\boxed{\frac{y^{-3}}{-3}}}_{=v} = -2x^5$$

$$\Longrightarrow v' - \frac{3}{x}v = -2x^5 \text{ Linear}$$

$$P(x) = -\frac{3}{x} \Longrightarrow I(x) = e^{-\int \frac{3}{x}dx} = e^{-3lnx} = e^{lnx^{-3}} = \frac{1}{x^3}$$

$$\Longrightarrow \frac{1}{x^3}v' - \frac{3}{x^4}v = -2x^2$$

$$\text{i.e. } (\frac{1}{x^3}v)' = -2x^2$$

$$\implies \frac{1}{x^3}v = -2\frac{x^3}{3} + C$$

$$\text{i.e. } \frac{1}{x^3}\frac{y^{-3}}{(-3)} = -2\frac{x^3}{3} + C$$

where C is an arbitrary constant.

Q3)

$$y' - \frac{1}{x}y = -\frac{y^2}{x}; \ \boxed{x < 0} \text{ Bernoulli}$$

$$y^{-2}y' - \frac{1}{x}y^{-1} = -\frac{1}{x}$$

$$\underbrace{\boxed{y^{-2}y'}}_{=v'} - \frac{1}{x}(-1)\underbrace{\boxed{\frac{y^{-1}}{-1}}}_{=v} = -\frac{1}{x}$$

$$\implies v' + \frac{1}{x}v = -\frac{1}{x} \text{ Linear}$$

$$P(x) = \frac{1}{x} \implies I(x) = e^{\int \frac{1}{x}dx} = e^{ln|x|} = |x| = -x$$

$$\implies -xv' - v = 1$$

$$\text{i.e. } [(-x)v]' = 1$$

$$\implies (-x)v = x + C$$

$$\text{i.e. } (-x)\frac{y^{-1}}{-1} = x + C$$

where C is an arbitrary constant.

Q4)

$$y' + \frac{y}{2x} = \frac{x}{y^3}; \ y(1) = 2. \text{ We will solve for } \boxed{x > 0}. \text{ Bernoulli}$$

$$y^3y' + \frac{1}{2x}y^4 = x$$

$$\underbrace{\boxed{y^3y'}}_{=v'} + \frac{1}{2x}(4)\underbrace{\boxed{\frac{y^4}{4}}}_{=v} = x$$

$$\implies v' + \frac{2}{x}v = x \text{ Linear}$$

$$P(x) = \frac{2}{x} \implies I(x) = e^{\int \frac{2}{x}dx} = e^{2lnx} = e^{lnx^2} = x^2$$

$$\implies x^2v' + 2xv = x^3$$

$$\text{i.e. } (x^2v)' = x^3$$

$$\implies x^2 v = \frac{x^4}{4} + C$$

$$\text{i.e. } x^2 \frac{y^4}{4} = \frac{x^4}{4} + C$$

$$y(1) = 2 \implies C = \frac{15}{4}$$

$$\implies x^2 \frac{y^4}{4} = \frac{x^4}{4} + \frac{15}{4}$$

Q5)

$$xy' + y = (xy)^{3/2};\ y(1) = 4. \text{ We will solve for } \boxed{x > 0}. \text{ Bernoulli}$$

$$y^{-3/2} y' + \frac{1}{x} y^{-1/2} = x^{1/2}$$

$$\underbrace{\boxed{y^{-3/2} y'}}_{=v'} + \frac{1}{x}(-1/2) \underbrace{\boxed{\frac{y^{-1/2}}{-1/2}}}_{=v} = x^{1/2}$$

$$\implies v' - \frac{1}{2x} v = x^{1/2} \text{ Linear}$$

$$P(x) = -\frac{1}{2x} \implies I(x) = e^{-\int \frac{1}{2x} dx} = e^{-\frac{1}{2} lnx} = e^{lnx^{-1/2}} = x^{-1/2} = \frac{1}{x^{1/2}}$$

$$\implies \frac{1}{x^{1/2}} v' - \frac{1}{2x^{3/2}} = 1$$

$$\text{i.e. } (\frac{1}{x^{1/2}} v)' = 1$$

$$\implies \frac{1}{x^{1/2}} v = x + C$$

$$\text{i.e. } \frac{1}{x^{1/2}} \frac{y^{-1/2}}{-1/2} = x + C$$

$$y(1) = 4 \implies C = -2$$

$$\implies \frac{1}{x^{1/2}} \frac{y^{-1/2}}{-1/2} = x - 2$$

Q6)

$$x^2 y' + xy = \frac{y^3}{x};\ y(1) = 1. \text{ We will solve for } \boxed{x > 0}. \text{ Bernoulli}$$

$$y^{-3} y' + \frac{1}{x} y^{-2} = \frac{1}{x^3}$$

$$\underbrace{\boxed{y^{-3} y'}}_{=v'} + \frac{1}{x}(-2) \underbrace{\boxed{\frac{y^{-2}}{-2}}}_{=v} = \frac{1}{x^3}$$

$$\implies v' - \frac{2}{x} v = \frac{1}{x^3} \text{ Linear}$$

$$P(x) = -\frac{2}{x} \Longrightarrow I(x) = e^{-\int \frac{2}{x}dx} = e^{-2lnx} = e^{lnx^{-2}} = \frac{1}{x^2}$$

$$\Longrightarrow \frac{1}{x^2}v' - \frac{2}{x^3}v = \frac{1}{x^5}$$

$$\text{i.e. } (\frac{1}{x^2}v)' = x^{-5}$$

$$\Longrightarrow \frac{1}{x^2}v = \frac{x^{-4}}{-4} + C$$

$$\text{i.e. } \frac{1}{x^2}\frac{y^{-2}}{(-2)} = \frac{x^{-4}}{-4} + C$$

$$y(1) = 1 \Longrightarrow C = -\frac{1}{4}$$

$$\Longrightarrow \frac{1}{x^2}\frac{y^{-2}}{-2} = \frac{x^{-4}}{-4} - \frac{1}{4}$$

Q7)

$$\underbrace{\boxed{(cosy)y'}}_{=v'} + \frac{1}{x}\underbrace{\boxed{siny}}_{=v} = 1;\ x > 0$$

$$\Longrightarrow v' + \frac{1}{x}v = 1 \text{ Linear}$$

$$P(x) = \frac{1}{x} \Longrightarrow I(x) = e^{\int \frac{1}{x}dx} = e^{lnx} = x$$

$$\Longrightarrow xv' + v = x$$

$$\text{i.e. } (xv)' = x$$

$$\Longrightarrow xv = \frac{x^2}{2} + C$$

$$\text{i.e. } x(siny) = \frac{x^2}{2} + C$$

where C is an arbitrary constant.

Q8)

$$(y+1)y' + x(y^2 + 2y) = x$$

$$\underbrace{\boxed{(y+1)y'}}_{=v'} + x(2)\underbrace{\boxed{\frac{y^2+2y}{2}}}_{=v} = x$$

$$\Longrightarrow v' + 2xv = x \text{ Linear}$$

$$P(x) = 2x \Longrightarrow I(x) = e^{\int 2xdx} = e^{x^2}$$

$$\Longrightarrow e^{x^2}v' + 2xe^{x^2}v = xe^{x^2}$$

$$\text{i.e. } (e^{x^2}v)' = xe^{x^2}$$

$$\Longrightarrow e^{x^2}v = \int xe^{x^2}dx + C$$

$$= \frac{1}{2}\int (2x)e^{x^2}dx + C$$

$$= \frac{1}{2}e^{x^2} + C$$

$$\text{i.e. } e^{x^2}(\frac{y^2+2y}{2}) = \frac{1}{2}e^{x^2} + C$$

where C is an arbitrary constant.

Q9)

$$\underbrace{\boxed{\frac{1}{y}y'}}_{=v'} + \frac{3}{x}\underbrace{\boxed{lny}}_{=v} = 6x^2;\ x > 0$$

$$\Longrightarrow v' + \frac{3}{x}v = 6x^2 \text{ Linear}$$

$$P(x) = \frac{3}{x} \Longrightarrow I(x) = e^{\int \frac{3}{x}dx} = e^{3lnx} = e^{lnx^3} = x^3$$

$$\Longrightarrow x^3v' + 3x^2v = 6x^5$$

$$\text{i.e. } (x^3v)' = 6x^5$$

$$\Longrightarrow x^3v = x^6 + C$$

$$\text{i.e. } x^3lny = x^6 + C$$

where C is an arbitrary constant.