H.W. 12 Solution

Q1)

$$x^2y'' - 2xy' + 2y = x^3;\ x > 0$$

$$x = e^t,\ D^k = \frac{d^k}{dx^k},\ \mathfrak{D}^k = \frac{d^k}{dt^k}$$

$$\Longrightarrow \mathfrak{D}(\mathfrak{D}-1)y - 2\mathfrak{D}y + 2y = e^{3t}$$

$$\underbrace{\mathfrak{D}^2y - 3\mathfrak{D}y + 2y}_{=f(\mathfrak{D})y} = \underbrace{e^{3t}}_{g(\mathfrak{D})=\mathfrak{D}-3}$$

$$\boxed{?y_c}\ f(m) = m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 2, 1$$

$$y_c = C_1e^{2t} + C_2e^t$$

$$\boxed{?y_p}\ g(\mathfrak{D})f(\mathfrak{D})y = 0$$

$$\overset{CH-EQ}{\Longrightarrow} (m-3)(m-2)(m-1) = 0$$

$$m = 3, 2, 1$$

$$y_{\bar{c}} = \alpha_1e^{3t} + \alpha_2e^{2t} + \alpha_3e^t$$

$$\overset{Try}{\Longrightarrow} y_p = Ae^{3t}$$

$$\Longrightarrow y_p' = 3Ae^{3t},\ \&$$

$$y_p'' = 9Ae^{3t}$$

$$\overset{DE}{\Longrightarrow} 9Ae^{3t} - 3(3Ae^{3t}) + 2Ae^{3t} = e^{3t}$$

$$\Longrightarrow A = \frac{1}{2}$$

$$y_p = \frac{1}{2}e^{3t}$$

$$\Longrightarrow \text{G. S. } y = y_c + y_p = C_1e^{2t} + C_2e^t + \frac{1}{2}e^{3t}$$

$$= C_1x^2 + C_2x + \frac{1}{2}x^3$$

Q2)

$$x^3y^{(3)} - 4x^2y^{(2)} + 8xy^{(1)} - 8y = 4lnx;\ x > 0$$

$$x = e^t,\ D^k = \frac{d^k}{dx^k},\ \mathfrak{D}^k = \frac{d^k}{dt^k}$$

$$\Longrightarrow \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)y - 4\mathfrak{D}(\mathfrak{D}-1)y + 8\mathfrak{D}y - 8y = 4t$$

$$\underbrace{(\mathfrak{D}^3 - 7\mathfrak{D}^2 + 14\mathfrak{D} - 9)}_{=f(\mathfrak{D})}y = \underbrace{4t}_{g(\mathfrak{D})=\mathfrak{D}^2}$$

$$\boxed{?y_c}\ f(m) = m^3 - 7m^2 + 14m - 8 = 0$$

$$(m-1)(m^2 - 6m + 8) = 0$$

$$(m-1)(m-4)(m-2) = 0$$

$$m = 1, 2, 4$$

$$y_c = C_1e^t + C_2e^{2t} + C_3e^{4t}$$

$$\boxed{?y_p}\ g(\mathfrak{D})f(\mathfrak{D})y = 0$$

$$\overset{CH-EQ}{\Longrightarrow} m^2(m-1)(m-4)(m-2) = 0$$

$$m = 0, 0, 1, 2, 4$$

$$y_{\bar{c}} = \alpha_1 + \alpha_2 t + \alpha_3 e^t + \alpha_4 e^{2t} + \alpha_5 e^{4t}$$

$$\overset{Try}{\Longrightarrow} y_p = \underset{?}{A} + \underset{?}{B}t$$

$$\Longrightarrow y_p' = B,\ y_p'' = 0 = y_p'''$$

$$\overset{DE}{\Longrightarrow} 14B - 8(A + Bt) = 4t$$

$$\Longrightarrow 14B - 8A = 0,\ -8B = 4$$

$$\Longrightarrow B = -1/2,\ A = -7/8$$

$$\Longrightarrow \text{G. S. } y = y_c + y_p = C_1e^t + C_2e^{2t} + C_3e^{4t} - \frac{7}{8} - \frac{1}{2}t$$

$$= C_1x + C_2x^2 + +C_3x^4 - \frac{7}{8} - \frac{1}{2}lnx$$

Q3)

$$x^3y^{(3)} - x^2y^{(2)} - 2xy^{(1)} - 4y = 0;\ x > 0$$

$$x = e^t,\ D^k = \frac{d^k}{dx^k},\ \mathfrak{D}^k = \frac{d^k}{dt^k}$$

$$\Longrightarrow \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)y - \mathfrak{D}(\mathfrak{D}-1)y - 2\mathfrak{D}y - 4y = 0$$

$$\underbrace{(\mathfrak{D}^3 - 4\mathfrak{D}^2 + \mathfrak{D} - 4)}_{=f(\mathfrak{D})}y = 0$$

$$f(m) = m^3 - 4m^2 + m - 4 = 0$$

$$m^2(m-4) + (m-4) = 0$$

$$(m-4)(m^2+1) = 0$$

$$m = 4, \pm i$$

$$\Longrightarrow \text{G. S. } y = C_1e^{4t} + C_2cost + C_3sint$$

$$= C_1x^4 + C_2cos(lnx) + C_3sin(lnx)$$

Q4)

$$2x^2y'' - 3xy' - 3y = 0 \text{ for } x \neq 0$$

$$\boxed{x > 0}\ x = e^t,\ D^k = \frac{d^k}{dx^k},\ \mathfrak{D}^k = \frac{d^k}{dt^k}$$

$$\Longrightarrow 2\mathfrak{D}(\mathfrak{D}-1)y - 3\mathfrak{D}y - 3y = 0$$

$$\underbrace{(2\mathfrak{D}^2 - 5\mathfrak{D} - 3)}_{=f(\mathfrak{D})}y = 0$$

$$f(m) = 2m^2 - 5m - 3 = 0$$

$$(2m+1)(m-3) = 0$$

$$m = -\frac{1}{2}, 3$$

$$\Longrightarrow \text{G. S. } y = C_1e^{-\frac{1}{2}t} + C_2e^{3t}$$

$$\boxed{= C_1x^{-\frac{1}{2}} + C_2x^3;\ x > 0}$$

$$\boxed{?\, x < 0}\ \text{ Let } \tilde{x} = -x \Longrightarrow \frac{dy}{dx} = -\frac{dy}{d\tilde{x}},\ \&\ \frac{d^2y}{dx^2} = \frac{d^2y}{d\tilde{x}^2}$$

$$\Longrightarrow 2(-\tilde{x})^2\frac{d^2y}{d\tilde{x}^2} - 3(-\tilde{x})(-\frac{dy}{d\tilde{x}}) - 3y = 0$$

$$\implies 2\tilde{x}^2\frac{d^2y}{d\tilde{x}^2} - 3\tilde{x}\frac{dy}{d\tilde{x}} - 3y = 0;\ \tilde{x} > 0$$

As above $y = C_1(\tilde{x})^{-\frac{1}{2}} + C_2(\tilde{x})^3;\ \tilde{x} > 0$

$$\boxed{y = C_1(-x)^{-\frac{1}{2}} + C_2x^3;\ x < 0}$$