H.W. 10 Solution

Q1)

$$(x^2+1)y'' - 2xy' + 2y = 0$$

$$y_1(x) = x$$

$$\text{Let } y_2(x) = v(x)x \quad \boxed{?v}$$

$$\Longrightarrow y_2' = v'x + v \quad \&$$

$$y_2'' = v''x + 2v'$$

$$\overset{DE}{\Longrightarrow} (x^2+1)[v''x + 2v'] - 2x[v'x + v] + 2vx = 0$$

$$x(x^2+1)\underbrace{\boxed{v''}}_{=z'} + 2\underbrace{\boxed{v'}}_{=z} = 0$$

$$\Longrightarrow z' + \frac{2}{x(x^2+1)}z = 0$$

$$\Longrightarrow \frac{z'}{z} = -\frac{2}{x(x^2+1)} = (-2)\{\frac{1}{x} - \frac{x}{x^2+1}\}$$

$$\Longrightarrow ln|z| = -2ln|x| + ln|x^2+1|$$

$$\Longrightarrow z = \frac{x^2+1}{x^2}$$

$$\Longrightarrow v' = \frac{x^2+1}{x^2} = 1 + \frac{1}{x^2}$$

$$\Longrightarrow v = x - \frac{1}{x}$$

$$\Longrightarrow y_2 = x^2 - 1$$

$$\Longrightarrow \text{G. S. } y = C_1 x + C_2(x^2-1)$$

Q2)

$$x^2y'' - 4xy' + 4y = 0; \; x > 0$$

$$y_1(x) = x$$

$$\text{Let } y_2(x) = vx \quad \boxed{?v}$$

$$\Longrightarrow y_2' = v'x + v \quad \&$$

$$y_2'' = v''x + 2v'$$

$$\overset{DE}{\Longrightarrow} x^2(v''x+2v') - 4x(v'x+v) + 4vx = 0$$

$$\Longrightarrow x^3v'' - 2x^2v' = 0$$

$$\Longrightarrow \underbrace{\boxed{v''}}_{=z'} - \frac{2}{x}\underbrace{\boxed{v'}}_{=z} = 0$$

$$\Longrightarrow \frac{z'}{z} = \frac{2}{x}$$

$$\Longrightarrow ln|z| = 2ln|x|$$

$$\Longrightarrow z = x^2$$

$$\Longrightarrow v' = x^2$$

$$\Longrightarrow v = \frac{x^3}{3}$$

$$\Longrightarrow y_2 = \frac{x^4}{3}$$

$$\Longrightarrow \text{G. S. } y = C_1x + C_2x^4$$

Q3)

$$y'' - y = e^x$$

$$y'' - y = 0(m^2 - 1 = 0, m = \pm 1) \Longrightarrow y_c = C_1\underbrace{\boxed{e^x}}_{=y_1} + C_2e^{-x}$$

$$\text{Let } y_p = v(x)e^x \quad \boxed{?v}$$

$$\Longrightarrow y_p' = v'e^x + ve^x \quad \&$$

$$y_p'' = v''e^x + 2v'e^x + ve^x$$

$$\overset{DE}{\Longrightarrow} (v''e^x + 2v'e^x + ve^x) - ve^x = e^x$$

$$\Longrightarrow \underbrace{\boxed{v''}}_{=z'} + 2\underbrace{\boxed{v'}}_{=z} = 1$$

$$\Longrightarrow z' + 2z = 1$$

$$\Longrightarrow z = \frac{1}{2}$$

$$\Longrightarrow v' = \frac{1}{2} \Longrightarrow v = \frac{1}{2}x$$

$$\Longrightarrow y_p = \frac{1}{2}xe^x$$

$$\Longrightarrow \text{G. S. } y = y_c + y_p = C_1 e^x + C_2 e^{-x} + \frac{1}{2} x e^x$$

Q4)

$$y'' + y = cosecx$$

$$y'' + y = 0 (m^2 + 1 = 0, m = \pm i) \Longrightarrow y_c = C_1 cosx + C_2 \underbrace{\boxed{sinx}}_{=y_1}$$

$$\text{Let } y_p = v(x) sinx \quad \boxed{?v}$$

$$\Longrightarrow y_p' = v' sinx + v cosx \quad \&$$

$$y_p'' = v'' sinx + 2v' cosx - v sinx$$

$$\overset{DE}{\Longrightarrow} (v'' sinx + 2v' cosx - v sinx) + v sinx = cosecx$$

$$\Longrightarrow \underbrace{\boxed{v''}}_{=z'} + \frac{2cosx}{sinx} \underbrace{\boxed{v'}}_{=z} = \frac{1}{sin^2 x}$$

$$P(x) = \frac{2cosx}{sinx} \Longrightarrow I(x) = e^{\int \frac{2cosx}{sinx} dx} = sin^2 x$$

$$\Longrightarrow (z sin^2 x) = 1 \Longrightarrow z sin^2 x = x$$

$$\Longrightarrow z = x cosec^2 x$$

$$\Longrightarrow v' = x cosec^2 x$$

$$\Longrightarrow v = \int x (cosec^2 x) dx$$

$$= \int x (-cotx)' dx$$

$$= -x cotx + \int cotx dx$$

$$= -x cotx + ln|sinx|$$

$$\Longrightarrow y_p = -x cosx + sinx ln|sinx|$$

$$\Longrightarrow \text{G. S. } y = y_c + y_p = C_1 cosx + C_2 sinx - x cosx + sinx ln|sinx|$$

Q5)

$$2x^2 y'' + x y' - y = x; \;\; x > 0$$

$$y_1(x) = x$$

$$\text{Let } y_2(x) = vx \quad \boxed{?v}$$

$$\Longrightarrow y_2' = v'x + v \quad \&$$

$$y_2'' = v''x + 2v'$$

$$2x^2y'' + xy' - y = 0 \Longleftrightarrow 2x^2(v''x + 2v') + x(v'x + v) - vx = 0$$

$$\Longrightarrow 2x^3 \underbrace{\boxed{v''}}_{=z'} + 5x^2 \underbrace{\boxed{v'}}_{=z} = 0$$

$$\Longrightarrow \frac{z'}{z} = -\frac{5}{2x}$$

$$\Longrightarrow ln|z| = -\frac{5}{2}ln|x|$$

$$\Longrightarrow z = x^{-5/2}$$

$$\Longrightarrow v' = x^{-5/2}$$

$$\Longrightarrow v = \frac{x^{-3/2}}{-3/2}$$

$$\Longrightarrow y_2 = -\frac{2}{3}x^{-1/2}$$

$$\Longrightarrow y_c = C_1x + C_2x^{-1/2}$$

$$\text{Let } y_p = w(x)x$$

$$\Longrightarrow y_p' = w'x + w \quad \&$$

$$y_p'' = w''x + 2w'$$

$$\overset{DE}{\Longrightarrow} 2x^2(w''x + 2w') + x(w'x + w) - wx = x$$

$$\Longrightarrow 2x^3 \underbrace{\boxed{w''}}_{=z'} + 5x^2 \underbrace{\boxed{w'}}_{=z} = x$$

$$\Longrightarrow z' + \frac{5}{2x}z = \frac{1}{2x^2}$$

$$\text{I.F} = e^{\int \frac{5}{2x}dx} = e^{\frac{5}{2}ln|x|} = x^{5/2}$$

$$\Longrightarrow (zx^{5/2})' = \frac{x^{1/2}}{2}$$

$$\Longrightarrow zx^{5/2} = \frac{x^{3/2}}{3}$$

$$\Longrightarrow w' = z = \frac{1}{3x}$$

$$\Longrightarrow w = \frac{1}{3}ln|x|$$

$$\Longrightarrow y_p = \frac{x}{3}lnx$$

$$\Longrightarrow \text{G. S. } y = y_c + y_p = C_1x + C_2x^{-1/2} + \frac{x}{3}lnx$$