H.W. 9 Solution

Q1)

$$y'' - 3y' + 2y = x^2 e^x$$

$$\underbrace{(D^2 - 3D + 2)}_{=f(D)}y = \underbrace{x^2 e^x}_{g(D)y=(D-1)^3 y=0}$$

Now y_p "lives" in $y_{\bar{c}} - y_c$, where $y_{\bar{c}}$ is the G. S. of $g(D)f(D)y = 0$ & y_c is the G. S. of $f(D)y = 0$.

$$\boxed{?y_c}\ (D^2 - 3D + 2)y = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$y_c = C_1 e^x + C_2 e^{2x}$$

$$\boxed{?y_{\bar{c}}}\ (D-1)^3(D^2 - 3D + 2)y = 0$$

$$(m-1)^4(m-2) = 0$$

$$m = 1, 1, 1, 1, 2$$

$$y_{\bar{c}} = \alpha_1 e^x + \alpha_2 x e^x + \alpha_3 x^2 e^x + \alpha_4 x^3 e^x + \alpha_5 e^{2x}$$

$$\Longrightarrow y_p = \{Ax + Bx^2 + Cx^3\}e^x$$

$$?A, B, C$$

$$\text{Now } y_p' = \{Ax + Bx^2 + Cx^3 + A + 2Bx + 3Cx^2\}e^x \ \&$$

$$y_p'' = \{Ax + Bx^2 + Cx^3 + A + 2Bx + 3Cx^2 + A + 2Bx + 3Cx^2 + 2B + 6Cx\}e^x$$

$$\overset{DE}{\Longrightarrow} \{Cx^3 + [6C + B]x^2 + [A + 4B + 6C]x + [2A + 2B]\}e^x - 3\{Cx^3 + [B + 3C]x^2 + [A + 2B]x + A\}e^x$$

$$+2\{Ax + Bx^2 + Cx^3\}e^x = x^2 e^x$$

$$\Longrightarrow 6C + B - 3B - 9C + 2B = 1$$

$$A + 4B + 6C - 3A - 6B + 2A = 0$$

$$2A + 2B - 3A = 0$$

$$\Longrightarrow C = \boxed{-1/3}, B = 3C = \boxed{-1}, A = 2B = \boxed{-2}$$

$$y_p = (-2x - x^2 - \frac{1}{3}x^3)e^x$$

$$\Longrightarrow \text{G. S. } y = y_c + y_p = C_1e^x + C_2e^{2x} + (-2x - x^2 - \frac{1}{3}x^3)e^x$$

Q2)

$$4y^{(3)} - 4y^{(2)} - 5y^{(1)} + 3y = 3x^3 - 8x$$

$$\underbrace{(4D^3 - 4D^2 - 5D + 3)}_{=f(D)}y = \underbrace{3x^3 - 8x}_{g(D)y = D^4y = 0}$$

Now $y_p \in y_{\bar{c}} - y_c$, where $y_{\bar{c}}$ is the G. S. of $g(D)f(D)y = 0$ & y_c is the G. S. of $f(D)y = 0$.

$$\boxed{?y_c}\ (4D^3 - 4D^2 - 5D + 3)y = 0$$

$$4m^3 - 4m^2 - 5m + 3 = 0$$

$$(m+1)(4m^2 - 8m + 3) = 0$$

$$(m+1)(2m-3)(2m-1) = 0$$

$$m = -1, \frac{3}{2}, \frac{1}{2}$$

$$y_c = C_1e^x + C_2e^{\frac{3}{2}x} + C_3e^{\frac{1}{2}x}$$

$$\boxed{?y_{\bar{c}}}\ D^4(4D^3 - 4D^2 - 5D + 3)y = 0$$

$$m^4(m+1)(2m-3)(2m-1) = 0$$

$$m = 0, 0, 0, 0, -1, \frac{3}{2}, \frac{1}{2}$$

$$y_{\bar{c}} = \alpha_1 + x\alpha_2 + x^2\alpha_3 + x^3\alpha_4 + \alpha_5e^{-x} + \alpha_6e^{\frac{3}{2}x} + \alpha_7e^{\frac{1}{2}x}$$

$$\Longrightarrow y_p = A + Bx + Cx^2 + Fx^3$$

$$?A, B, C, F$$

$$\text{Now } y_p' = B + 2Cx + 3Fx^2,$$

$$y_p'' = 2C + 6Fx, \ \&$$

$$y_p''' = 6F$$

$$\overset{DE}{\Longrightarrow} 4(6F) - 4(2C + 6Fx) - 5(B + 2Cx + 3Fx^2) + 3(A + Bx + Cx^2 + Fx^3) = 3x^3 - 8x$$

$$\implies 3F = 3$$

$$-15F + 3C = 0$$

$$-24F - 10C + 3B = -8$$

$$24F - 8C - 5B + 3A = 0$$

$$\implies F = \boxed{1}, C = \boxed{5}, B = \boxed{22}, A = \boxed{42}$$

$$y_p = 42 + 22x + 5x^2 + x^3$$

$$\implies \text{G. S. } y = y_c + y_p = C_1 e^{-x} + C_2 e^{\frac{3}{2}x} + C_3 e^{\frac{1}{2}x} + (42 + 22x + 5x^2 + x^3)$$

Q3)

$$y'' - y = 3x^2 e^x,\ y(0) = 1,\ y'(0) = 2$$

$$\underbrace{(D^2 - 1)}_{=f(D)} y = \underbrace{3x^2 e^x}_{g(D)y=(D-1)^3 y=0}$$

Now $y_p \in y_{\bar{c}} - y_c$, where $y_{\bar{c}}$ is the G. S. of $g(D)f(D)y = 0$ & y_c is the G. S. of $f(D)y = 0$.

$$\boxed{?y_c}\ (D^2 - 1)y = 0$$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$y_c = C_1 e^x + C_2 e^{-x}$$

$$\boxed{?y_{\bar{c}}}\ (D - 1)^3 (D^2 - 1) y = 0$$

$$(m - 1)^4 (m + 1) = 0$$

$$m = 1, 1, 1, 1, -1$$

$$y_{\bar{c}} = \alpha_1 e^x + \alpha_2 x e^x + \alpha_3 x^2 e^x + \alpha_4 x^3 e^x + \alpha_5 e^{-x}$$

$$\implies y_p = (Ax + Bx^2 + Cx^3) e^x$$

$$?A, B, C$$

$$\text{Now } y_p' = (Ax + Bx^2 + Cx^3 + A + 2Bx + 3Cx^2) e^x \ \&$$

$$y_p'' = (Ax + Bx^2 + Cx^3 + A + 2Bx + 3Cx^2 + A + 2Bx + 3Cx^2 + 2B + 6Cx) e^x$$

$$\overset{DE}{\Longrightarrow} (Ax+Bx^2+Cx^3+A+2Bx+3Cx^2+A+2Bx+3Cx^2+2B+6Cx)e^x-(Ax+Bx^2+Cx^3)e^x = 3x^2e^x$$

$$\Longrightarrow 6C = 3$$

$$4B + 6C = 0$$

$$2A + 2B = 0$$

$$\Longrightarrow C = \boxed{1/2}, B = \boxed{-3/4}, A = \boxed{3/4}$$

$$y_p = (\frac{3}{4}x - \frac{3}{4}x^2 + \frac{1}{2}x^3)e^x$$

$$\Longrightarrow \text{G. S. } y = y_c + y_p = C_1e^x + C_2e^{-x} + (\frac{3}{4}x - \frac{3}{4}x^2 + \frac{1}{2}x^3)e^x$$

$$\text{Now } y(0) = 1 \Longrightarrow C_1 + C_2 = 1$$

$$y'(0) = 2 \Longrightarrow C_1 - C_2 + \frac{3}{4} = 2$$

$$\Longrightarrow 2C_1 = 3 - \frac{3}{4} = \frac{9}{4} \Longrightarrow C_1 = \frac{9}{8}$$

$$\Longrightarrow C_2 = 1 - \frac{9}{8} = -\frac{1}{8}$$

$$\Longrightarrow \text{ The solution of the I.V.P. is}$$

$$y = \frac{9}{8}e^x - \frac{1}{8}e^{-x} + (\frac{3}{4}x - \frac{3}{4}x^2 + \frac{1}{2}x^3)e^x$$

Q4)

$$y''' - y' = e^x$$

$$\underbrace{(D^3 - D)}_{=f(D)}y = \underbrace{e^x}_{g(D)y=(D-1)y=0}$$

Now $y_p \in y_{\bar{c}} - y_c$, where $y_{\bar{c}}$ is the G. S. of $g(D)f(D)y = 0$ & y_c is the G. S. of $f(D)y = 0$.

$$\boxed{?y_c}\ (D^3 - D)y = 0$$

$$m^3 - m = 0$$

$$m(m^2 - 1) = 0$$

$$m = 0, 1, -1$$

$$y_c = C_1 + C_2e^x + C_3e^{-x}$$

$$\boxed{?y_{\bar{c}}}\ (D - 1)(D^3 - D)y = 0$$

$$(m-1)m(m^2-1)=0$$

$$(m-1)^2m(m+1)=0$$

$$m=1,1,0,-1$$

$$y_{\bar{c}}=\alpha_1e^x+\alpha_2xe^x+\alpha_3+\alpha_4e^{-x}$$

$$\Longrightarrow y_p=Axe^x$$

$$?A$$

$$\text{Now } y_p'=A(x+1)e^x,$$

$$y_p''=A(x+2)e^x,\ \&$$

$$y_p'''=A(x+3)e^x$$

$$\overset{DE}{\Longrightarrow} A(x+3)e^x-A(x+1)e^x=e^x$$

$$\Longrightarrow 2A=1\Longrightarrow A=1/2$$

$$y_p=\frac{1}{2}xe^x$$

$$\Longrightarrow \text{G. S. } y=y_c+y_p=C_1+C_2e^x+C_3e^{-x}+\frac{1}{2}xe^x$$

Q5)

$$y^{(4)}-2y^{(2)}+y=e^x$$

$$\underbrace{(D^4-2D^2+1)}_{=f(D)}y=\underbrace{e^x}_{g(D)y=(D-1)y=0}$$

Now $y_p\in y_{\bar{c}}-y_c$, where $y_{\bar{c}}$ is the G. S. of $g(D)f(D)y=0$ & y_c is the G. S. of $f(D)y=0$.

$$\boxed{?y_c}\ (D^4-2D^2+1)y=0$$

$$m^4-2m^2+1=0$$

$$(m^2-1)^2=0$$

$$(m-1)^2(m+1)^2=0$$

$$m=1,1,-1,-1$$

$$y_c=C_1e^x+C_2e^{-x}+C_3xe^x+C_4xe^{-x}$$

$$\boxed{?y_{\bar{c}}}\ (D-1)(D^4-2D^2+1)y=0$$

$$(m-1)(m^4-2m^2+1)=0$$

$$(m-1)^3(m+1)^2=0$$

$$m=1,1,1,-1,-1$$

$$y_{\bar{c}}=\alpha_1e^x+\alpha_2xe^x+\alpha_3x^2e^x+\alpha_4e^{-x}+\alpha_5xe^{-x}$$

$$\Longrightarrow y_p=Ax^2e^x$$

$$?A$$

$$\text{Now } y_p'=A(x^2+2x)e^x,$$

$$y_p''=A(x^2+2x+2x+2)e^x,$$

$$y_p'''=A(x^2+4x+2+2x+4)e^x,\ \&$$

$$y_p'''=A(x^2+6x+6+2x+6)e^x$$

$$\overset{DE}{\Longrightarrow} A(x^2+8x+12)e^x-2A(x^2+4x+2)e^x+Ax^2e^x=e^x$$

$$\Longrightarrow 8A=1\Longrightarrow A=\boxed{1/8}$$

$$y_p=\frac{1}{8}x^2e^x$$

$$\Longrightarrow \text{G. S. } y=y_c+y_p=C_1e^x+C_2e^{-x}+C_3xe^x+C_4xe^{-x}+\frac{1}{8}x^2e^x$$

Q6)

$$y''-2y'-3y=2e^x-10sinx$$

$$\underbrace{(D^2-2D-3)}_{=f(D)}y=\underbrace{2e^x-10sinx}_{g(D)y=(D-1)(D^2+1)y=0}$$

Now y_p "lives" in $y_{\bar{c}}-y_c,$ where $y_{\bar{c}}$ is the G. S. of $g(D)f(D)y=0$ & y_c is the G. S. of $f(D)y=0$.

$$\boxed{?y_c}\ (D^2-2D-3)y=0$$

$$m^2-2m-3=0$$

$$(m-3)(m+1)=0$$

$$m=3,-1$$

$$y_c = C_1e^{3x} + C_2e^{-x}$$

$$\boxed{?y_{\bar{c}}}\ (D-1)(D^2+1)(D^2-2D-3)y = 0$$

$$(m-1)(m^2+1)(m^2-2m-3) = 0$$

$$(m-1)(m^2+1)(m+1)(m-3) = 0$$

$$m = 1, \pm i, -1, 3$$

$$y_{\bar{c}} = \alpha_1 e^x + \alpha_2 cosx + \alpha_3 sinx + \alpha_4 e^{-x} + \alpha_5 e^{3x}$$

$$\Longrightarrow y_p = Ae^x + Bcosx + Csinx$$

$$?A, B, C$$

Now $y_p' = Ae^x - Bsinx + Ccosx$ &

$$y_p'' = Ae^x - Bcosx - Csinx$$

$$\overset{DE}{\Longrightarrow} (Ae^x - Bcosx - Csinx) - 2(Ae^x - Bsinx + Ccosx) - 3(Ae^x + Bcosx + Csinx) = 2e^x - 10sinx$$

$$\Longrightarrow -4A = 2$$

$$-B - 2C - 3B = 0 (\Longrightarrow 2B + C = 0)$$

$$-C + 2B - 3C = -10 (\Longrightarrow 2B - 4C = -10)$$

$$\Longrightarrow A = \boxed{-1/2}, C = \boxed{2}, B = \boxed{-1}$$

$$y_p = -\frac{1}{2}e^x - cosx + 2sinx$$

$$\Longrightarrow \text{G. S. } y = y_c + y_p = C_1e^{3x} + C_2e^{-x} - \frac{1}{2}e^x - cosx + 2sinx$$