H.W. 2 Solution

Q1)

$$xy' + (x+1)y = x^3; \ \boxed{x > 0}$$

$$y' = -\frac{x+1}{x}y + x^2 \text{ Linear}$$

$$y' + \frac{x+1}{x}y = x^2$$

$$P(x) = 1 + \frac{1}{x}$$

$$I(x) = e^{x+lnx} = e^x e^{lnx} = xe^x$$

$$xe^x y' + (x+1)e^x y = x^3 e^x$$

$$(xe^x y)' = x^3 e^x$$

$$xe^x y = \int x^3 e^x dx + C$$

Now

$$\int x^3 e^x dx = \int x^3 (e^x)' dx = x^3 e^x - \int e^x (3x^2) dx$$

$$= x^3 e^x - 3\int x^2 (e^x)' dx$$

$$= x^3 e^x - 3[x^2 e^x - \int e^x (2x) dx]$$

$$= x^3 e^x - 3x^2 e^x + 6\int x (e^x)' dx$$

$$= x^3 e^x - 3x^2 e^x + 6[xe^x - \int e^x .1 dx]$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x$$

$\Longrightarrow$ General solution for $\boxed{x > 0}$

$$xe^x y = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$$

or

$$y = x^2 - 3x + 6 - \frac{6}{x} + \frac{C}{x} e^{-x}$$

where C is an arbitrary constant.

Q2)

$$y' + \frac{2x+1}{x}y = e^{-2x};\ \boxed{x > 0}$$

$$y' = -\frac{2x+1}{x}y + e^{-2x} \text{ Linear}$$

$$y' + \frac{2x+1}{x}y = e^{-2x}$$

$$P(x) = 2 + \frac{1}{x}$$

$$I(x) = e^{2x+lnx} = e^{2x}e^{lnx} = xe^{2x}$$

$$y'xe^{2x} + (2x+1)e^{2x}y = x$$

$$(yxe^{2x})' = x$$

$$yxe^{2x} = \frac{x^2}{2} + C$$

where C is an arbitrary constant.

Q3)

$$y' + \frac{3y}{x} = 6x^2;\ \boxed{x < 0}$$

$$y' = -\frac{3}{x}y + 6x^2 \text{ Linear}$$

$$y' + \frac{3}{x}y = 6x^2$$

$$P(x) = \frac{3}{x}$$

$$I(x) = e^{\int \frac{3}{x}dx} = e^{3ln|x|} = e^{ln|x|^3} = |x|^3 = -x^3(\text{ Since } x < 0)$$

$$y'(-x^3) - 3x^2y = -6x^5$$

$$[y(-x^3)]' = -6x^5$$

$$y(-x^3) = -x^6 + C$$

where C is an arbitrary constant.

Q4)

$$y' + 3y = 3x^2e^{-3x}$$

$$y' = -3y + 3x^2e^{-3x} \text{ Linear}$$

$$y' + 3y = 3x^2e^{-3x}$$

$$P(x) = 3$$

$$I(x) = e^{3x}$$

$$e^{3x}y' + 3e^{3x}y = 3x^2$$

$$(ye^{3x})' = 3x^2$$

$$ye^{3x} = x^3 + C$$

where C is an arbitrary constant.

Q5)

$$\frac{dr}{d\theta} + rtan\theta = cos^2\theta;\ r(\frac{\pi}{4}) = 1$$

$$\frac{dr}{d\theta} = (-tan\theta)r + cos^2\theta \text{ Linear}$$

$$\frac{dr}{d\theta} + (tan\theta)r = cos^2\theta$$

$$P(\theta) = tan\theta = \frac{sin\theta}{cos\theta}$$

$$I(\theta) = e^{-ln(cos\theta)} = e^{ln(cos\theta)^{-1}} = \frac{1}{cos\theta}$$

Note $cos\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\frac{1}{cos\theta}\frac{dr}{d\theta} + \frac{sin\theta}{cos^2\theta}r = cos\theta$$

$$(\frac{1}{cos\theta}r)' = cos\theta$$

$$\frac{1}{cos\theta}r = sin\theta + C$$

$$r(\frac{\pi}{4}) = 1 \implies \frac{1}{\frac{1}{\sqrt{2}}} \cdot 1 = \frac{1}{\sqrt{2}} + C \implies C = \sqrt{2} - \frac{1}{\sqrt{2}}$$

$$\implies \frac{1}{cos\theta}r = sin\theta + \sqrt{2} - \frac{1}{\sqrt{2}}$$

Q6)

$$y' + y = f(x);\ y(0) = 6$$

$$\text{where} f(x) = \begin{cases} 5 & ;\quad 0 \le x < 10 \\ 1 & ;\quad x \ge 10 \end{cases}$$

$$y' = (-1)y + f(x) \text{ Linear}$$

$$y' + (1)y = f(x)$$

$$P(x) = 1$$

$$I(x) = e^x$$

$$e^x y' + ye^x = f(x)e^x$$

$$(e^x y)' = f(x)e^x$$

$$\implies e^x y - e^0 y(0) = \int_0^x f(z)e^z dz$$

$$\text{i.e. } e^x y - 6 = \int_0^x f(z)e^z dz$$

$$\text{i.e. } e^x y = 6 + \int_0^x f(z)e^z dz$$

Now for $0 \leq x < 10$

$$e^x y = 6 + \int_0^x 5e^z dz = 6 + 5[e^x - 1] = 1 + 5e^x$$

$$\implies y = e^{-x} + 5$$

For $x \geq 10$

$$e^x y = 6 + \int_0^{10} 5e^z dz + \int_{10}^x 1 \cdot e^z dz = 6 + 5[e^{10} - 1] + [e^x - e^{10}] = 1 + e^x + 4e^{10}$$

$$\implies y = e^{-x} + 1 + 4e^{10-x}$$

$$\implies \text{Solution } y(x) = \begin{cases} e^{-x} + 5 & ; \quad 0 \leq x < 10 \\ e^{-x} + 1 + 4e^{10-x} & ; \quad x \geq 10 \end{cases}$$