H.W. 13 Solution

Q1)

$$y(t) = cost + \int_0^t e^{-z} y(t-z)dz$$

$$= cost + y(t) * e^{-t}$$

$$\xrightarrow[\pounds]{} Y = \frac{s}{1+s^2} + Y\frac{1}{s+1}$$

$$\Longrightarrow \{1 - \frac{1}{s+1}\}Y = \frac{s}{s^2+1}$$

$$\Longrightarrow \{\frac{s}{s+1}\}Y = \frac{s}{s^2+1}$$

$$\Longrightarrow Y = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{y(t) = cost + sint}$$

Q2)

$$y'(t) + 6y + 9\int_0^t y(z)dz = 1;\ y(0) = 0$$

$$\xrightarrow[\pounds]{} \{sY - 0\} + 6Y + 9\frac{Y}{s} = \frac{1}{s}$$

$$\Longrightarrow s^2Y + 6sY + 9Y = 1$$

$$\Longrightarrow Y = \frac{1}{s^2+6s+9} = \frac{1}{(s+3)^2}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{y(t) = e^{-3t}t}$$

Q3)

$$y'' - 6y' + 9y = t^2e^{3t};\ y(0) = 2,\ y'(0) = 6$$

$$\xrightarrow[\pounds]{} \{s^2Y - s\cdot 2 - 6\} - 6\{sY - 2\} + 9Y = \frac{2!}{(s-3)^3}$$

$$\Longrightarrow (\underbrace{s^2 - 6s + 9}_{=(s-3)^2})Y = 2s - 6 + \frac{2}{(s-3)^3}$$

$$= 2(s-3) + \frac{2}{(s-3)^3}$$

$$\Longrightarrow Y = \frac{2}{s-3} + \frac{2}{(s-3)^5}$$

$$= \frac{2}{s-3} + (\frac{2}{4!})\frac{4!}{(s-3)^5}$$

$$\xrightarrow[\pounds^{-1}]{} y(t) = 2e^{3t} + (\frac{2}{4!})e^{3t}t^4$$

$$\Longrightarrow \boxed{y(t) = 2e^{3t} + \frac{1}{12}e^{3t}t^4}$$

Q4)

$$y'' + 4y = 1\{u_t(\pi) - u_t(2\pi)\}$$

$$\xrightarrow[\pounds]{} \{s^2Y - s \cdot 1 - 0\} + 4Y = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$\Longrightarrow Y = \frac{s}{s^2+4} + \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-2\pi s}}{s(s^2+4)}........(*)$$

$$\text{Now } \pounds^{-1}[\frac{1}{s(s^2+4)}] = \pounds^{-1}[\frac{1}{4}(\frac{1}{s} - \frac{s}{s^2+4})]$$

$$= \frac{1}{4} - \frac{1}{4}cos(2t)$$

$$\xrightarrow[\pounds^{-1}(*)]{} y(t) = cos2t + [\frac{1}{4} - \frac{1}{4}cos2(t-\pi)]u_t(\pi) - [\frac{1}{4} - \frac{1}{4}cos2(t-2\pi)]u_t(2\pi)$$

Q5)

$$x' = 6x + y$$

$$y' = 4x + 3y$$

$$x(0) = 2,\ y(0) = 7$$

$$\xrightarrow[\pounds]{}$$

$$\left.\begin{aligned}(sX - 2) &= 6X + Y \\ (sY - 7) &= 4X + 3Y\end{aligned}\right\}$$

$$\Longrightarrow$$

$$\left.\begin{aligned}(s-6)X - Y &= 2 \\ -4X + (s-3)Y &= 7\end{aligned}\right\}.............(*)$$

$$\overset{(*)}{\Longrightarrow} \{(s-3)(s-6) - 4\}X = 2(s-3) + 7$$

$$\Longrightarrow X = \frac{2s+1}{s^2 - 9s + 14}$$

$$= \frac{2s+1}{(s-7)(s-2)}$$

$$= \frac{3}{s-7} + \frac{-1}{s-2}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{x(t) = 3e^{7t} - e^{2t}}$$

$$\text{Also } (*) \implies \{(s-6)(s-3)-4\}Y = 8 + 7(s-6)$$

$$\implies Y = \frac{7s-34}{s^2-9s+14}$$

$$= \frac{7s-34}{(s-7)(s-2)}$$

$$= \frac{3}{s-7} + \frac{4}{s-2}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{y(t) = 3e^{7t} + 4e^{2t}}$$

Q6)

$$x'' + 16x = cos4t\{u_t(0) - u_t(\pi)\}$$

$$= cos4t\{1 - u_t(\pi)\}$$

$$= cos4t - cos4(t-\pi)u_t(\pi)$$

$$\xrightarrow[\pounds]{} \{s^2X(s) - s\cdot 0 - 1\} + 16X(s) = \frac{s}{s^2+16} - e^{-\pi s}\frac{s}{s^2+16}$$

$$\implies X(s) = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - e^{-\pi s}\frac{s}{(s^2+16)^2}$$

$$\text{Now } \pounds^{-1}[\frac{s}{(s^2+16)^2}] = \pounds^{-1}[\frac{s}{s^2+16}\cdot\frac{1}{4}\frac{4}{s^2+16}]$$

$$= \frac{1}{4}cos4t * sin4t$$

$$= \frac{1}{4}\int_0^t cos4(t-z)sin4zdz$$

$$= \frac{1}{8}\int_0^t [sin4t + sin(8z-4t)]dz$$

$$= \frac{1}{8}(sin4t)t + \{-\frac{cos(8z-4t)}{8}\}|_0^t]$$

$$= \boxed{\frac{1}{8}[(sin4t)t} - \frac{1}{64}\{cos4t - \underbrace{cos(-4t)}_{=0}\}$$

$$\text{(You can also use (11) on the table to get } \pounds^{-1}[\frac{s}{(s^2+16)^2}]$$

$$= \pounds^{-1}[\frac{1}{8}\frac{8s}{(s^2+16)^2}]$$

$$= \frac{1}{8}tsin4t)$$

$$\xrightarrow[\pounds^{-1}]{} x(t) = \frac{1}{4}sin4t + \frac{1}{8}tsin4t + \frac{1}{8}(t-\pi)sin4(t-\pi)u_t(\pi)$$

Q7)

$$x' = x + 3y$$

$$y' = 3x + y$$

$$x(0) = 1,\ y(0) = 0$$

$$\xrightarrow[\pounds]{}$$

$$\left.\begin{aligned}(sX-1) &= X + 3Y\\ (sY-0) &= 3X + Y\end{aligned}\right\}$$

$$\Longrightarrow$$

$$\left.\begin{aligned}(s-1)X - 3Y &= 1\\ -3X + (s-1)Y &= 0\end{aligned}\right\}.............(*)$$

$$\overset{(*)}{\Longrightarrow} \{(s-1)^2 - 9\}X = s - 1$$

$$\Longrightarrow X = \frac{s-1}{(s-1)^2 - 9}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{x(t) = e^t cosh3t}$$

$$\text{Also } (*) \Longrightarrow \{(s-1)^2 - 9\}Y = 3$$

$$\Longrightarrow Y = \frac{3}{(s-1)^2 - 9}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{y(t) = e^t sinh3t}$$

Q8)

$$x' = 3x - 3y$$

$$y' = 6x - 3y$$

$$x(0) = 4,\ y(0) = 3$$

$$\xrightarrow[\pounds]{}$$

$$\left.\begin{aligned}(sX-4) &= 3X - 3Y\\ (sY-3) &= 6X - 3Y\end{aligned}\right\}$$

$$\Longrightarrow$$

$$\left.\begin{aligned} (s-3)X + 3Y &= 4 \\ -6X + (s+3)Y &= 3 \end{aligned}\right\} \text{.............}(*)$$

$$\overset{(*)}{\Longrightarrow} \{(s^2-9)+18\}X = 4(s+3) - 9$$

$$\Longrightarrow X = \frac{4s+3}{s^2+9}$$

$$= 4\left(\frac{s}{s^2+9}\right) + \frac{3}{s^2+9}$$

$$\underset{\pounds^{-1}}{\longrightarrow} \boxed{x(t) = 4cos(3t) + sin(3t)}$$

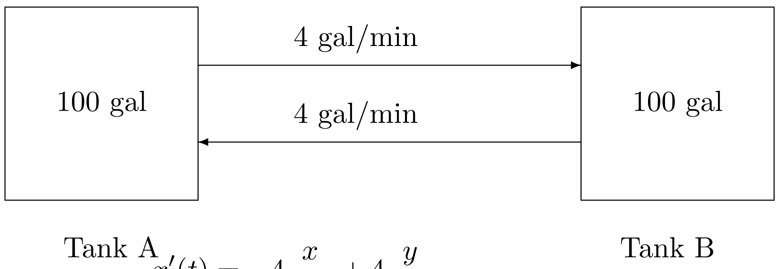
$$\text{Also } (*) \Longrightarrow \{(s^2-9)+18\}Y = 24 + 3(s-3)$$

$$\Longrightarrow Y = \frac{3s+15}{s^2+9}$$

$$= 3\left(\frac{s}{s^2+9}\right) + 5\left(\frac{3}{s^2+9}\right)$$

$$\underset{\pounds^{-1}}{\longrightarrow} \boxed{y(t) = 3cos(3t) + 5sin(3t)}$$

Q9)



$$x'(t) = -4\frac{x}{100} + 4\frac{y}{100}$$

$$y'(t) = 4\frac{x}{100} - 4\frac{y}{100}$$

$$x(0) = 8,\ y(0) = 4$$

$$\underset{\pounds}{\longrightarrow}$$

$$\left.\begin{aligned} (sX-8) &= -\tfrac{X}{25} + \tfrac{Y}{25} \\ (sY-4) &= \tfrac{X}{25} - \tfrac{Y}{25} \end{aligned}\right\}$$

$$\Longrightarrow$$

$$\left.\begin{aligned} (s+\tfrac{1}{25})X - \tfrac{Y}{25} &= 8 \\ -\tfrac{X}{25} + (s+\tfrac{1}{25})Y &= 4 \end{aligned}\right\} \text{.............}(*)$$

$$\overset{(*)}{\Longrightarrow} \{(s+\frac{1}{25})^2-(\frac{1}{25})^2\}X = 8(s+\frac{1}{25})+\frac{4}{25}$$

$$\Longrightarrow X = \frac{8(s+\frac{1}{25})}{(s+\frac{1}{25})^2-(\frac{1}{25})^2}+\frac{4(\frac{1}{25})}{(s+\frac{1}{25})^2-(\frac{1}{25})^2}$$

$$\underset{\pounds^{-1}}{\longrightarrow} x(t) = 8e^{-\frac{t}{25}}cosh(\frac{t}{25})+4e^{-\frac{t}{25}}sinh(\frac{t}{25})$$

$$= 4\{1+e^{-\frac{2t}{25}}\}+2\{1-e^{-\frac{2t}{25}}\}$$

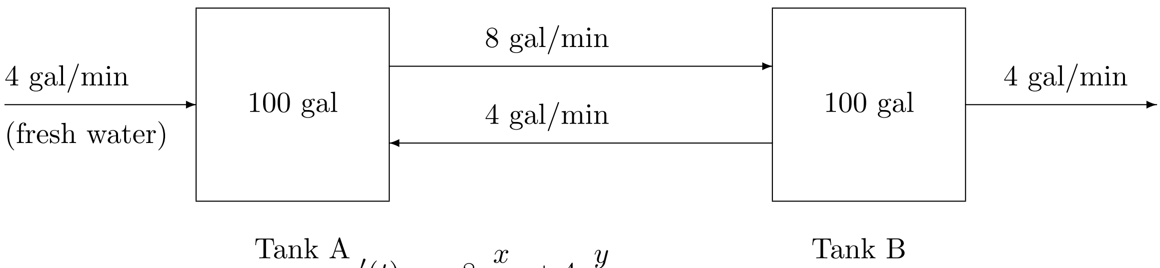
$$\boxed{x(t) = 6+2e^{-\frac{2t}{25}}}$$

$$\Longrightarrow \boxed{y(t) = 6-2e^{-\frac{2t}{25}}} \text{ (since } y(t) = 12-x(t))$$

? When will the amount of chemical in Tank B be 6 1bs i.e. when will $y(t) = 6$.

Answer: Never

Q10)



$$x'(t) = -8\frac{x}{100}+4\frac{y}{100}$$

$$y'(t) = 8\frac{x}{100}-8\frac{y}{100}$$

$$x(0) = 0,\ y(0) = 10$$

$$\underset{\pounds}{\longrightarrow}$$

$$\left.\begin{aligned}(sX-0) &= -8\tfrac{X}{100}+4\tfrac{Y}{100}\\ (sY-10) &= 8\tfrac{X}{100}-8\tfrac{Y}{100}\end{aligned}\right\}$$

$$\Longrightarrow$$

$$\left.\begin{aligned}(s+\tfrac{8}{100})X-4\tfrac{Y}{100} &= 0\\ -\tfrac{8}{100}X+(s+\tfrac{8}{100})Y &= 10\end{aligned}\right\}\ldots\ldots\ldots\ldots(*)$$

$$\overset{(*)}{\Longrightarrow} \{(s+\frac{8}{100})^2-\frac{32}{(100)^2}\}X = \frac{40}{100}$$

$$\Longrightarrow X = \frac{(40/100)}{(s+\frac{8}{100})^2 - \frac{32}{(100)^2}}$$

$$= \frac{40}{\sqrt{32}} \frac{(\frac{\sqrt{32}}{100})}{(s+\frac{8}{100})^2 - (\frac{\sqrt{32}}{100})^2}$$

$$\xrightarrow[\pounds^{-1}]{} x(t) = \frac{40}{\sqrt{32}} e^{-\frac{8}{100}t} sinh(\frac{\sqrt{32}}{100}t)$$

$$= \frac{40}{\sqrt{32}} e^{-\frac{8}{100}t} \{\frac{e^{\frac{\sqrt{32}}{100}t} - e^{-\frac{\sqrt{32}}{100}t}}{2}\}$$

$$= \frac{20}{\sqrt{32}} \{e^{-(\frac{8-\sqrt{32}}{100})t} - e^{-(\frac{8+\sqrt{32}}{100})t}\}$$

$$\overset{(*)}{\Longrightarrow} \{(s+\frac{8}{100})^2 - \frac{32}{(100)^2}\} Y = 10(s+\frac{8}{100})$$

$$\Longrightarrow Y = \frac{10(s+\frac{8}{100})}{(s+\frac{8}{100})^2 - (\frac{\sqrt{32}}{100})^2}$$

$$\xrightarrow[\pounds^{-1}]{} y(t) = 10 e^{-\frac{8}{100}t} cosh(\frac{\sqrt{32}}{100}t)$$

$$= 5\{e^{-(\frac{8-\sqrt{32}}{100})t} + e^{-(\frac{8+\sqrt{32}}{100})t}\}$$

Note: as $t \to \infty$ both $x(t)$ and $y(t) \to 0$.

Q11)

$$y'' + 3y' + 2y = 1[u_t(0) - u_t(2)] - 1[u_t(2) - u_t(4)]$$

$$= 1 - 2u_t(2) + u_t(4)]$$

$$\xrightarrow[\pounds]{} [s^2 Y - s \cdot 0 - 0] + 3[sY - 0] + 2Y = \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-4s}}{s}$$

$$\Longrightarrow Y = \frac{1}{\underbrace{s(s^2+3s+2)}_{=s(s+2)(s+1)}} - \frac{2e^{-2s}}{s(s^2+3s+2)} + \frac{e^{-4s}}{s(s^2+3s+2)}$$

$$\text{Now } \pounds^{-1}[\frac{1}{s(s+1)(s+2)}] = \pounds^{-1}[\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}]$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

$$\Longrightarrow y(t) = [\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}] - 2[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}]u_t(2) + [\frac{1}{2} - e^{-(t-4)} + \frac{1}{2}e^{-2(t-4)}]u_t(4)$$

Q12)

$$y'' + y = (sint)u_t(2\pi) = sin(t-2\pi)u_t(2\pi)$$

$$\xrightarrow[\pounds]{} [s^2Y - s\cdot 0 - 0] + Y = e^{-2\pi s}\frac{1}{s^2+1}$$

$$\Longrightarrow Y = e^{-2\pi s}\frac{1}{(s^2+1)^2}$$

$$\text{Now } \pounds^{-1}[\frac{1}{(s^2+1)^2}]$$

$$= sint * sint$$

$$= \int_0^t sin(t-z)sinzdz$$

$$= \frac{1}{2}\int_0^t [cos(t-2z) - cost]dz$$

$$= \frac{1}{2}[\frac{sin(t-2z)}{-2}|_0^t - (cost)t]$$

$$= -\frac{1}{4}[(-sint) - sint] - \frac{1}{2}tcost$$

$$= \frac{1}{2}[sint - tcost]$$

$$\xrightarrow[\pounds^{-1}]{} y(t) = \frac{1}{2}[sin(t-2\pi) - (t-2\pi)cos(t-2\pi)]u_t(2\pi)$$

Q13)

$$y'' + 5y' + 4y = 0;\ y(0) = 1,\ y'(0) = 0$$

$$\xrightarrow[\pounds]{} \{s^2Y - s\cdot 1 - 0\} + 5\{sY - 1\} + 4Y = 0$$

$$\Longrightarrow Y = \frac{s+5}{s^2+5s+4} = \frac{s+5}{(s+4)(s+1)}$$

$$= \frac{(-1/3)}{s+4} + \frac{4/3}{s+1}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{y(t) = -\frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t}}$$

Q14)

$$y'' - 6y' + 9y = t;\ y(0) = 0,\ y'(0) = 1$$

$$\xrightarrow[\pounds]{} \{s^2Y - s\cdot 0 - 1\} - 6\{sY - 0\} + 9Y = \frac{1}{s^2}$$

$$\Longrightarrow (s^2 - 6s + 9)Y = 1 + \frac{1}{s^2}$$

$$= \frac{s^2+1}{s^2}$$

$$\Longrightarrow Y = \frac{s^2+1}{s^2(s-3)^2}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$

$$\Longrightarrow s^2+1 = As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2$$

$$\text{Let } \boxed{s=0} \Longrightarrow 1 = 9B \Longrightarrow \boxed{B = 1/9}$$

$$\text{Let } \boxed{s=3} \Longrightarrow 10 = 9D \Longrightarrow \boxed{D = 10/9}$$

$$\boxed{\text{"}s^3\text{"}} \Longrightarrow 0 = A + C$$

$$\boxed{\text{"}s\text{"}} \Longrightarrow 0 = 9A - 6B \Longrightarrow \boxed{A = \frac{2}{3}B = \frac{2}{27}} \text{ then } \boxed{C = -\frac{2}{27}}$$

$$\Longrightarrow Y = \frac{(2/27)}{s} + \frac{(1/9)}{s^2} + \frac{(-2/27)}{s-3} + \frac{(10/9)}{(s-3)^2}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{y(t) = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}e^{3t}t}$$

Q15)

$$2y''' + 3y'' - 3y' - 2y = e^{-t};\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1$$

$$\xrightarrow[\pounds]{} 2[s^3Y - s^2 \cdot 0 - s \cdot 0 - 1] + 3[s^2Y - s \cdot 0 - 0] - 3[sY - 0] - 2Y = \frac{1}{s+1}$$

$$\Longrightarrow (2s^3 + 3s^2 - 3s - 2)Y = 2 + \frac{1}{s+1}$$

$$\Longrightarrow (s-1)(2s^2 + 5s + 2)Y = \frac{2s+3}{s+1}$$

$$\text{i.e. } (s-1)(2s+1)(s+2)Y = \frac{2s+3}{s+1}$$

$$\Longrightarrow Y = \frac{2s+3}{(s-1)(2s+1)(s+2)(s+1)}$$

$$= \frac{(5/18)}{s-1} + \frac{(-16/9)}{2s+1} + \frac{(1/9)}{s+2} + \frac{(1/2)}{s+1}$$

$$= \frac{(5/18)}{s-1} - \frac{(8/9)}{s+\frac{1}{2}} + \frac{(1/9)}{s+2} + \frac{(1/2)}{s+1}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{y(t) = \frac{5}{18}e^t - \frac{8}{9}e^{-\frac{1}{2}t} + \frac{1}{9}e^{-2t} + \frac{1}{2}e^{-t}}$$

Q16)

$$f(t) + \int_0^t (t-\tau)f(\tau)d\tau = t$$

$$\text{i.e. } f(t) + t * f(t) = t$$

$$\xrightarrow[\pounds]{} F(s) + \frac{F(s)}{s^2} = \frac{1}{s^2}$$

$$\Longrightarrow (1+\frac{1}{s^2})F(s) = \frac{1}{s^2}$$

$$\Longrightarrow F(s) = \frac{1}{s^2+1}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{f(t) = sint}$$

Q17)

$$f(t) = 1 + t - \frac{8}{3}\int_0^t (\tau - t)^3 f(\tau) d\tau$$

$$= 1 + t + \frac{8}{3}\int_0^t (t-\tau)^3 f(\tau) d\tau$$

$$= 1 + t + \frac{8}{3}t^3 * f(t)$$

$$\xrightarrow[\pounds]{} F(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{8}{3}F(s)\frac{3!}{s^4}$$

$$\Longrightarrow (1 - \frac{16}{s^4})F(s) = \frac{1}{s} + \frac{1}{s^2} = \frac{s+1}{s^2}$$

$$\Longrightarrow (\frac{s^4-16}{s^4})F(s) = \frac{s+1}{s^2}$$

$$\Longrightarrow F(s) = \frac{s^2(s+1)}{s^4-16} = \frac{s^2(s+1)}{(s^2+4)(s-2)(s+2)}$$

$$= \frac{As+B}{s^2+4} + \frac{C}{s-2} + \frac{D}{s+2}$$

$$\Longrightarrow s^2(s+1) = (As+B)(s^2-4) + C(s+2)(s^2+4) + D(s-2)(s^2+4)$$

$$\text{Let } \boxed{s=2} \Longrightarrow 4 \cdot 3 = C(4)(8) \Longrightarrow \boxed{C = 3/8}$$

$$\text{Let } \boxed{s=-2} \Longrightarrow 4(-1) = D(-4)(8) \Longrightarrow \boxed{D = 1/8}$$

$$\boxed{"s^3"} \Longrightarrow 1 = A + C + D$$

$$\Longrightarrow A = 1 - (C+D) = 1 - (3/8 + 1/8) = 1 - 4/8 = \boxed{1/2}$$

$$\text{Let } \boxed{s=0} \Longrightarrow 0 = -4B + 8C - 8D$$

$$\Longrightarrow B = 2C - 2D = 2(3/8) - 2(1/8) = 4/8 = \boxed{1/2}$$

$$\Longrightarrow F(s) = \frac{(1/2)s + 1/2}{s^2+4} + \frac{3/8}{s-2} + \frac{1/8}{s+2}$$

$$= (\frac{1}{2})\frac{s}{s^2+4} + (\frac{1}{2})\frac{1}{s^2+4} + \frac{3/8}{s-2} + \frac{1/8}{s+2}$$

$$\xrightarrow[\pounds^{-1}]{} \boxed{f(t) = \frac{1}{2}(cos2t) + \frac{1}{2}(sin2t) + \frac{3}{8}e^{2t} + \frac{1}{8}e^{-2t}}$$

Q18)

$$f(t) + \int_0^t f(\tau)d\tau = 1$$

$$\xrightarrow[\mathcal{L}]{} F(s) + \frac{F(s)}{s} = \frac{1}{s}$$

$$\implies (1 + \frac{1}{s})F(s) = \frac{1}{s}$$

$$\implies F(s) = \frac{1}{s+1}$$

$$\xrightarrow[\mathcal{L}^{-1}]{} \boxed{f(t) = e^{-t}}$$